THE DISTRIBUTION OF ZEROS OF SOLUTIONS OF SIXTH - ORDER DIFFERENTIAL EQUATION WITH A MEASURABLE COEFFICIENTS

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Abstract

In this paper, the issue of distribution of zeros of the solutions of linear differential equations (LDE) have been investigated in terms of semi - critical intervals. We shall follow a geometric approach to state and prove some properties of LDEs of the sixth order boundary conditions and with measurable coefficients. Moreover, the relations between semi - critical intervals of the LDEs have been explored. Also, the obtained results have been generalized for the 5^th order differential equations.

Key words:

Linear differential equations; distribution of zeros for the solution; boundary value problems; semi-oscillatory interval; semi-critical interval..

1 INTRODUCTION

laws of distribution of zeros of solutions began to emerge in some studies,

considering not only one differential equation and multi - point boundary value problems for the given equation, but more generally, multi - point boundary value problems for any given equation.

The question of the laws of distribution of zeros of solutions of a linear differential equation touches upon many studies on the theory and practice of differential equations.

Mikusinsky. [20] The following analogue of the Sturm theorem[1] is obtained:

If the solutions u (t) and v (t) of the equation $x^{(n)}$+ g (t) x = 0 satisfy the conditions

$$u(\alpha) = u'(\alpha) = \cdots\cdots = u^{(n-2)}(\alpha) = 0,\ u^{(n-1)}(\alpha) = 1,\ u(\beta) = 0,$$

$v(\gamma) = v'(\gamma) = \cdots\cdots = v^{(n-2)}(\gamma) = 0$, $v^{(n-1)}(\gamma) = 1$, $\alpha < \gamma < \beta$, then the solution v (t) does not have zeros in (□, □]. A. K. Kondratiev. [18], considering the same equation $x^{(n)} + g(t)x = 0$ for values of n = 3 and n = 4 at constant coefficient $g(t)$, proved the following theorem on alternation of zeros of solutions:

1) if n = 3, then between two consecutive zeros of one solution there are at most two zeros of the other;

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[1]If t_1,t_2 are successive zeros of the solution x_1(t) of the second-order equation with continuous coefficients $x'' + g_0(t)x' + g_1(t)x = 0$, then every other linearly independent solution $x_2(t)$ has exactly one zero between t_1 and t_2.

2) if n = 4 and g (t) > 0, then between two consecutive zeros of one solution lies not more than four zeros of the other, where four zeros can lie;

3) if n = 4 and g (t) <0, then between two successive zeros of one solution lie not more than three zeros of the other.

A.M. Akhundov. and A. T. Toraev. [11] found a generalization of the result of Kondratiev.for the equation $x''' + g_1(t)x' + g_2(t)x = 0$, where $g_1(t) < 0$, $g_2(t)$ is constant-sign.

U. Levin [19] showed that the theorem of A. K. Kondratiev is also valid for equations of the form $x''' + g_1(t)x'' + g_2(t)x' = 0$, and $x^{(IV)} + (g(t)x')' = 0$.

After the publication of [12], where the laws of distribution of zeros of solutions for a LHDE of the third order of the general form for n = 3 in terms of semi - critical intervals established, a large number of papers have appeared [1-3, 12, 15, 21]. They studied with one or other degrees of completeness of the problem of the distribution of zeros of solutions of equations of nth order at n □ 4 with summable coefficients besides the continuous ones.

The authors of [4-10, 16 and 17] investigated LDEs of the (fifth, sixth) order, they used the analytic approach to prove the properties of the distribution zeros of their solutions.

In this paper, we shall rather use the geometric approach to state and prove some properties of LDE of the sixth order with (2 points) boundary conditions.

Main results in this study are;

- If $r_{42}(\alpha) \neq r_{51}(\alpha)$ ($r_{15}(\alpha) < r_{24}(\alpha)$) then there is not (non-trivial) solution $u(x)$ of equation (2.1) satisfying the boundary

$u^{(i)}(t_1) = u(t_2) = u(t_3) = 0$ ($u(t_1) = u(t_2) = u^{(i)}(t_3) = 0$),

$$i = 0,1,2,3,\ \alpha \le t_1 < t_2 < t_3 = r_{411}(\alpha).$$

- Two linearly independent solutions $u(t)$,$v(t)$ of equation (1) with a common 4 - multiple zero t_1 have no more common zeros in $[\alpha, r_{51}(\alpha))$ and

$[\alpha, r_{42}(\alpha))$ ($[\alpha, r_{15}(\alpha))$. and $[\alpha, r_{24}(\alpha))$) on the right (left) of t_1.

2 CONCEPTS AND TERMINOLOGY CONSIDER THE EQUATION

$$L[y] = x^{(6)} - \sum_{j=0}^{5} g_j(t)x^{(j)} = 0, \qquad (2.1)$$

Assume that the coefficients $g_k(x)$ are measurable and continuous on $[a, b]$ satisfying the conditions

$$x_j^{(k_j)}(t_j) = A_{j,k},\ k_j = 0, \dots\dots, p_j - 1, j = 1, 2, \dots\dots, m, \sum_{j=0}^{m} p_j = 6, m \le 6 \quad (2.2)$$

where m is the number of points t_j, p_j is the number of conditions at the points t_j.

Problem (2.1) and (2.2) is called ≪ ($p_1p_2 \dots p_m$ – problem) ≫.

Definition 2.1 [9]: For each fixed point $\alpha \in [a, b)$, there exists a nonzero interval $[\alpha, \beta)$, in which any non - trivial solution of equation (2.1) has no more than 5 zero, taking into account their multiplicities. This interval is called the semi - oscillation for equation (2.1). The maximum intervals of semi - oscillation with a common origin in α is denoted by $[\alpha, r(\alpha))$.

Definition 2.2 [8]: The interval $[\alpha, \mu)$, in which the given problem has a unique solution, is called the semi - critical interval of this problem. The maximum intervals of semi - critical with a common origin in α is denoted by

$[\alpha, r_{p_1p_2\dots p_k}(\alpha))$, k =2, 3, 4, 5.

The concept of the semi - critical interval is directly related to the distribution of zeros of the solution of equation (2.1).

We decipher the definitions of the maximal semi - critical intervals of some boundary value problems.

The interval $[\alpha, r_{51}(\alpha))$ is called such an interval in which any non - trivial solution (for the equation (2.1)) that has a zero at t_1 of multiplicity five and has no more zeros to the right of t_1, where $\alpha \le t_1 < r_{51}(s) < t_2$.

In the interval $[\alpha, r_{42}(\alpha))$, non - trivial solution (for the equation (2.1)) that has a zero at t_1 of multiplicity four, can not have a double zero to the right of t_1, where $\alpha \le t_1 < r_{42}(\alpha) < t_2$.

A Non - trivial solution (for the equation (2.1)) that has a zero at $t_1$of multiplicity three and zero $t_2 > t_1$ can not have a zero $t_3 > t_2$ of multiplicity higher than the second in the interval $[\alpha, r_{312}(\alpha))$, where $\alpha \le t_1 < t_2 < r_{312}(\alpha) < t_3$.

In the interval $[\alpha, r_{111111}(\alpha))$ a nontrivial solution can not have six different simple zeros.

Let us clarify justifications of listed assertions. On the example of the problem

≪(42 − problem)≫ to have a unique solution, It is necessary and sufficient that the determinant

$u_0(t_1)\ u_1(t_1)\ u_2(t_1)\ u_3(t_1)\ u_4(t_1)\ u_5(t_1)$

u'

|𝑢′′′′0(((𝑡𝑡𝑡111)))𝑢𝑢𝑢1′′′1′′1′ (((𝑡𝑡𝑡111)))𝑢𝑢𝑢2′′′2′′2′ (((𝑡𝑡𝑡111)))𝑢𝑢𝑢3′′′3′′3′ (((𝑡𝑡𝑡111)))𝑢𝑢𝑢4′′′4′′4′ (((𝑡𝑡𝑡111))) 𝑢𝑢𝑢55′′′′5′ (((𝑡𝑡𝑡111)))| =△ (𝑡1, 𝑡2) ≠ 0,

0

$u0$

|

$u_0(t_2)\ u_1(t_2)\ u_2(t_2)\ u_3(t_2)\ u_4(t_2)\ u_5(t_2)$

$u0'(t2)\ u1'(t2)\ u2'(t2)\ u3'(t2)\ u4'(t2)\ u5'(t2)$ where $u_i(t_1)$, i = 0, 1, 2, 3, 4, is the fundamental system of solutions of equa-tion (2.1), was different from zero. But $\Delta(t_1, x)$ is a solution of equation (2.1), and at the point t_1 this solution has a four - multiple zero. Thus, the condition does not vanish (is not vanishing) at $x \in [t_1, \omega)$.

The Function $\Delta(t_1, x)$ is a condition for the existence and uniqueness of the solution of problem (2.1), (2.2) for any $t_2 \in [t_1, \omega)$.

In a similar way, one can make sure of other cases.

Consider the following auxiliary lemmas.

Lemma 2.1 [6]: Let $v_1(t)$, $v_2(t)$ be a pair of not identically equal to zero, twice continuously differentiable functions such that

$v_1(t) \ne cv_2(t)$, (c = $const$), $v_1(\alpha) = v_1(\beta) = 0$, $v_2(t) \ne 0$ and $v_1(t)v_2(t) >$
0 in $[\alpha, \beta]$.

Then there exists a linear combination

$$x(t) = c_1v_1(t) + c_2v_2(t), (c_1^2 + c_2^2 > 0),$$

for which the point ξ is a zero of multiplicity two, that is

$$x(\xi) = x'(\xi) = 0, \quad \text{where } \xi \in (\alpha, \beta).$$

Lemma 2.2 [6]: Let $v_1(t), v_2(t)$ – be a pair of not identically equal to zero,
thrice continuously differentiable functions such that,

$$v_1(t) \ne cv_2(t), (\text{c} = const), v_1^{(k)}(t) = v_2^{(k)}(t) = 0, k = 0, 1.$$

Then, there exists a linear combination

$$x(t) = c_1v_1(t) + c_2v_2(t), (c_1^2 + c_2^2 > 0),$$

for which the point ξ is a zero of multiplicity three, that is

$$x^{(i)}(\xi) = 0, i = 0, 1, 2.$$

Lemma 2.3 [7]: Non - trivial solutions $v_1(t)$ and $v_2(t)$ of equation (1) are linearly dependent if

$$v_i^{(k)}(\xi) = 0, k = 0, 1, 2; \; i = 1, 2.$$

Lemma 2.4 [7]: Let $u(t), v(t)$ – be a pair of non-trivial solutions of equation (1) such that $u^{(k)}(\alpha) = 0, k = 0, 1, 2, 3, 4; \; v(\alpha) = 0$. If $u(t) \neq 0$ in $(\alpha, \beta + \varepsilon)$, then for any $\varepsilon > 0$ and for some constant c, the difference $cu(t) - v(t)$ vanishes (goes to zero) at the points $\beta_i \in (\alpha, \beta + \varepsilon)$, whose number is equal to $p + q$, where p is the number of odd zeros of the solution $v(x)$ in $(\alpha, \beta]$ and q is the number of those $\alpha_i \in (\alpha, \beta]$, that

$$v(\alpha_i) = v'(\alpha_i) = 0, \; u'''(\alpha)v''(\alpha_i) > 0.$$

3 MAIN RESULTS

In this section, according to the mentioned conditions in the definitions above, we will prove the following theorems

Theorem 3.1:
If $r_{42}(\alpha) \neq r_{51}(\alpha)$, then there is not (non-trivial) solution $u(x)$ of equation (2.1) satisfying the boundary

$$u(t_1) = u'(t_1) = u''(t_1) = u'''(t_1) = u(t_2) = u(t_3) = 0,$$

$$\alpha \leq t_1 < t_2 < t_3 = r_{411}(\alpha). \qquad (3.1)$$

Proof:

Under the conditions of the theorem, let solution $u(t)$ is exist that satisfies conditions (3.1).
We set: $r_{51}(\alpha) < r_{42}(\alpha)$. Assume that, the solution $v(x)$ of equation (2.1) has two zeros t_1, β, such that

$$v^{(i)}(t_1) = 0, i = 0, 1, 2, 3, 4; \; v(\beta) = 0. \qquad (3.2)$$

It can be verified that $u^{(4)}(t_1)v^{(5)}(t_1) < 0$.
Then

$$u(t)\,v(t) > 0, t_2 < t < t_3.$$

And therefore, if $\beta > t_3$, then to a non-trivial linear combination

$$x(t) = u(t) - c\,v(t)$$

applies Lemma 2.1 it has double zero

$$\boldsymbol{\gamma} \in (t_2, t_3) \subset [\,\boldsymbol{\alpha}, r_{51}(\alpha)).$$

If $\beta > t_3$, then regarding c $= \frac{u'(t_3)}{v'(t_3)}$, $u'(t_3)v'(t_3) \neq 0$, since $r_{51}(\alpha) < r_{42}(\alpha)$, solution $x(t)$ has double zero at $\beta = t_3$.

But because of the conditions of (3.1), (3.2) the solution $x(t)$ has another 4- multiple zero at point t_1.
And this contradicts the definition of the interval $[\alpha, r_{42}(\alpha))$.
let now $r_{411}(\alpha) = r_{42}(\alpha) < r_{51}(\alpha)$. consider solution $v(t)$ of equation (2.1) such that

$$v^{(i)}(t_1) = v(\beta) = v'(\beta) = 0, i = 0, 1, 2, 3, \qquad (3.3)$$

If $\beta = t_3$, then because of the conditions (3.1), (3.3) non-trivial linear combination

$$x(t) = u(t) - c\,v(t)$$

in the interval $[\alpha, r_{51}(\alpha))$ has zero at the point t_3 and 5- multiple zero at point t_1 Where

$$\boldsymbol{c} = \frac{u^{(4)}(t_1)}{v^{(4)}(t_1)}$$

$u^{(4)}(t_1)\, v^{(4)}(t_1) \neq 0$, since $r_{42}(\alpha) < r_{51}(\alpha)$, this contradicts the definition of the interval $[\alpha, r_{51}(\alpha))$.

If $\beta > t_3$, then believing $u^{(4)}(t_1)v^{(4)}(t_1) < 0$, by Lemma (2.1), there exists a point $\gamma \in (t_2, t_1)$ such that $x(\gamma) = x'(\gamma) = 0$.

Hereby, by virtue of conditions (3,1),(3.3) the solution $x(t)$ has another 4 - multiple zero at point t_1, contrary to the definition of the interval$[\alpha, r_{42}(\alpha))$. The theorem is proved.

Theorem 3.2:

let $r_{15}(\alpha) < r_{24}(\alpha)$, then there is not (non-trivial) solution $u(x)$ of equation (2.1) satisfying the boundary

$$u(t_1) = u(t_2) = u(t_3) = u'(t_3) = u''(t_3) = u'''(t_3) = 0,$$
$$\alpha \le t_1 < t_2 < t_3 = r_{114}(\alpha), \qquad (3.4)$$

Proof:

let solution $u(t)$exists that satisfies conditions (3.4).

We set **:**

$$r_{15}(\alpha) < r_{24}(\alpha).$$

By setting that, solution $v(t)$ of equation (2.1), such that

$$v(\sigma) = v(t_3) = v'(t_3) = v''(t_3) = v'''(t_3) = v^{(4)}(t_3) = 0 \qquad (3.5)$$

we can assume that $u^{(4)}(t_3)v^{(5)}(t_3) < 0$.

Then

$$u(t)\, v(t) > 0,\ t_1 < t < t_2.$$

If $\sigma < t_1$, then the difference

$$x(t) = u(t) - c\, v(t)$$

applies Lemma 2.1 the difference has double zero

$$\boldsymbol{\gamma} \in (t_1, t_2) \subset [\, \boldsymbol{\alpha}, r_{24}(\alpha)).$$

If

$\sigma < t_1$ then regording $c = \dfrac{u'(t_1)}{v'(t_1)}, u'(t_1)v'(t_1) \neq 0$, since $r_{15}(\alpha) < r_{24}(\alpha)$, solution $x(t)$ has double zero at $\sigma = t_1$.

But because of the conditions of (3.4), (3.5) the solution $x(t)$ has 4- multiple zero at point t_3.

This can not verifies in the interval $[\alpha, r_{24}(\alpha))$.

let $r_{114}(\alpha) = r_{24}(\alpha) < r_{15}(\alpha)$. Suppose solution $v(t)$ of equation (2.1) which obeys

$$v(\sigma) = v'(\sigma) = v(t_3) = v'(t_3) = v''(t_3) = v'''(t_3) = 0 \qquad (3.6)$$

If $\sigma = t_1$, then because of the conditions (3.4), (3.6), the difference

$$x(t) = u(t) - c\, v(t)$$

in the interval $[\alpha, r_{51}(\alpha))$ has zero at the point t_1 and 5- multiple zero at point t_3 Where

$\boldsymbol{c} = \dfrac{u^{(4)}(t_3)}{v^{(4)}(t_3)}, u^{(4)}(t_3)\, v^{(4)}(t_4) \neq 0$, since $r_{24}(\alpha) < r_{15}(\alpha)$,

this contradicts the definition of the interval $[\alpha, r_{51}(\alpha))$.

If $\sigma < t_1$, then believing $u^{(4)}(t_3)v^{(4)}(t_3) < 0$, by Lemma 2.1, there exists a point $\gamma \in (t_1, t_2)$ such that $x(\gamma) = x'(\gamma) = 0$.

Hereby, by virtue of conditions (3.4),(3.6) the solution $x(t)$ has 4 - multiple zero at point t_3,

This can not happen inthe interval $[\alpha, r_{24}(\alpha))$. The theorem is proved.

Theorem 3.3.

Two linearly independent solutions $u(t)$, $v(t)$ of equation (2.1) with a common 4 - multiple zero $t_1 \in [\alpha, \rho(\alpha))$ have no more common zeros in $[\alpha, r_{51}(\alpha))$ and $[\alpha, r_{42}(\alpha))$ ($[\alpha, r_{42}(\alpha))$ and $[\alpha, r_{42}(\alpha))$) on the right (left) of t_1.

Proof:

Let the theorem be false and t_0 common zero of $u(t)$ and $v(t)$ in the specified intervals, where t_0 different from t_1.

If

$$\alpha \le t_1 < t_0 < r_{42}(\alpha) \ (\ \alpha \le t_0 < t_1 < r_{24}(\alpha)\ )$$

then the solution of the form

$$x(t) = u(t) - \frac{u'(t_0)}{v'(t_0)}\, v(t)$$

has 4 - multiple zero t_1 and double zero $t_0 > t_1$ ($t_0 < t_1$) , which is impossible. where

$$t_0, t_1 \in [\alpha, r_{42}(\alpha)) \ ([\alpha, r_{42}(\alpha))$$

If
$t_0 \in [\alpha, r_{51}(\alpha))$ ($[\alpha, r_{15}(\alpha)$) and $t_0 > t_1$ ($t_0 < t_1$),
then the solution

$$x(t) = u(t) - \boldsymbol{c}\ v(t)$$

Where c = const would have 5 - multiple zero t_1 and zero $t_0 > t_1$ ($t_0 < t_1$), $\alpha \le t_1 < t_0 < r_{51}(\alpha)$ ($\alpha \le t_0 < t_1 < r_{15}(\alpha)$), which is impossible. The theorem is proved

4 CONCLUSIONS

This study is an investigation of the distribution of zeros of non - trivial solutions of a linear homogeneous differential equation of sixth order in terms of semi - critical intervals of boundary value problems. It also includes the description of the behavior trend of the estimated intervals of uniqueness of the solutions.

Basically, we have obtained new results (Theorems 3.1, 3.2, 3.3,). Using these theorems, we have established the limiting relations between the lengths of semi - critical intervals of the uniqueness of solutions of two points boundary value problems with fixed points and the description of their estimated behavior.

Competing Interests

Author has declared that no competing interests exist.

References

[1] R. G. Aliev, S. h Gamidov. Boundary value problems for ordinary differential equations. Educational allowance, Makhachkala.1992; 78.

[2] R. G. Aliev. On the question of distribution of zeros of solutions of linear differential equations of fourth order. Recnts.of Sci.Asp.Conf., Kazan Inst; 1962.

[3] R. G. Aliev. On some properties of solutions of the fourth-order ordinary differential equation. News of universities. Mathematics. 1965; (45) 2:3-6.

[4] S. A. S. Al-Joufi. Limit relations between the semi-critical intervals solutions of multipoint boundary value problems ofor linear fifth-order differential equation. Herald of Dagestan State University (DSU). 2012; 6: 61–66.

[5]] S. A. S. Al-Joufi. On the distribution of zeros of solutions and their first derivatives of the fifth order linear differential equation. V All-Russian scientific - practical conference. "The scientific initiative of foreign students and graduate students of Russian universities" Collection of reports. (April 25-27). Tomsk. 2012; 1: 316-318.

[6] S. A. S. Al-Joufi. On the relations between the intervals of uniqueness of solutions of boundary value problems for a fifth-order equation. Actual problems of mathematics and related questions (materials of the International Conference "Mukhtar Readings", (DSTU) . Makhachkala.2012; 19-23.

[7] S. A. S. Al-Joufi, A. H. Kathim. The laws of the distribution of zeros of solutions of a linear differential equation of order 5. Herald of Dagestan State University (DSU). 2012; 1: 79–86.

[8] S. A. S. Al-Joufi. On the zeros of solutions of linear fifth-order differential equation. Actual problems of mathematics and related questions (materials of the International Conference "Mukhtar Readings", (DSTU). Makhachkala.2012; 23-26.

[9] S. A. S. Al-Joufi.On the laws of the distribution of zeros of non - trivial solutions of a linear differential equation in the terminology of conjugate simi - critical intervals. Functional differential equations and their applications (materials of the VI International Scientific Conference, dedicated to the 80th anniversary of the DSU).(September 23-26) Makhachkala.2013; 41-47.

[10] S. A. S. Al-Joufi. Method transfer to the limit, for resolve the problems of non - Vallée - Poussin type. Actual problems of mathematics and related questions; (DSTU). Makhachkala.2017: 11-19.

[11] A. M. Akhundov, A. T. Toraev. On solutions of the differential equation. Known. AN Turkm. SSR, ser. fizmatite. techn. chemical. and geol sciences. 1964; 1: 21-23.

[12] N. V. Azbelev, B. Tsalyuk. On the problem of the distribution of the zeros of solutions of a fourth-order linear differential equation. Mateem. Sat. 1960; 51(4): 475-486.

[13] A. Faroq. Baker, A. Hassan. El-Morshedy. The distribution of zeros of all solutions of first order neutral differential equations.Applied Mathematics and Computation. New York,USA. 2015; 259: 777-789.

[14] L. Hui, H. Zhenlai , S. Shurong . The distribution of zeros of oscillatory solutions for second order nonlinear neutral delay differential equations. Applied Mathematics Letters.2017;63:14-20.

[15] I. N. Inozemtseva.On the distribution of zeros of solutions of a linear differential equation of order n. Pro.Tikhnich Math. Tambov.1968; 2: 27-29.

[16] A. H. Kathim. The Distribution of the Solution Zeros for the Boundary Problems of Four Points of Linear Homogeneous Differential Equations of the Sixth Order. British Journal of Mathematics & Computer Science.2016;18(6):1-13.

[17] A. H. Kathim, Wafaa M. Taha. The distribution of zeros of the solutions of linear homogeneous differential equations of the sixth order using semi - critical intervals. Tikrit Journal of Pure Science. 2016; 21(3): 162-165.

[18] A. K. Kondratiev. On oscillation of the solutions of the equation $y^{(n)} + p(x)y = 0$. Proceedings of Moscow. Math. ob-va.1961; 10: 419-436.

[19] A. U. Levin. On the distribution of zeros of solutions of a linear differential equation. //DAN. 1964; 156(6): 1281-1284.

[20] I. Mikusinski,Sur lequation $X^{(n)}+A(t)X=0$, Ann.Polon.Math.1955; 1(26): 207-221.

[21] A. C. Peterson. A theorem of Aliev. Proc.Amer.Math.1969; 23(2): 364
–366.

[22] H. Samir. Saker, A. Mohammed. Arahet. Distributions of Zeros of Solutions for Third Order Differential Equations with Variable Coefficients. Mathematical Problems in Engineering 2015; Volume 2015: 1-9.

[23] H. Wu, L. Erbe. On the distance between consecutive zeros of solutions of first order delay differential equations, Appl.Math.Comput. 2013; 219: 8622–
8631.