EXAMPLE OF APPLICATION OF THE PONTRYAGIN’S MINIMUM PRINCIPLE “PMP EXTENSION”:
ZERMELO PROBLEM (WITH CURRENT SPEED MORE THAN BOAT SPEED HYPOTHESIS)

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Abstract. We present an example of application covering several cases using the extension of the Pontryagin minimum principle (PMP) in the case where we add a constraint on reaching a target variety at the final time: the Zermelo problem with current speed more than Boat speed hypothesis, where we consider a boat crossing a channel under a strong current and where we try to reach the opposite bank by minimizing the lateral offset or by minimizing the crossing time.

Пример применения принципа минимума Понтрягина «расширение PMP»: задача Zermelo (при гипотезе скорости течения больше, чем скорости лодки)

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Definition: Non-linear control systems
We consider the non-linear control system
\[
\dot{x}_u(t) = f(t, x_u(t), u(t)), \forall t \in [0, T], x_u(0) = x_0
\]
• \(x_u : [0, T] \to \mathbb{R}^d, T > 0 \) fixed , \(x_0 \in \mathbb{R}^d \) fixed
• \(u : [0, T] \to U \subset \mathbb{R}^k\), a closed non-empty sub-assembly
• \(f : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d\)

We are looking for an optimal control \(\tilde{u} \in U = L^1([0, T]; U)\) which minimizes the criterion
\[
J(u) = \int_0^T g(t, x_u(t), u(t)) \, dt + h(x_u(T))
\]

With \(g : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d\) and \(h : \mathbb{R}^d \to \mathbb{R}\)

With the usual known hypotheses on \(f, g\) and \(h\)

The theorem of the Pontryagin’s minimum principle PMP:
If \(\tilde{u} \in U\) is an optimal control, then by noting \(\tilde{x} = x_{\tilde{u}} \in AC ([0, T]; \mathbb{R}^d)\) the trajectory associated with the control \(\tilde{u}\) and by defining the deputy state \(\tilde{p} \in AC ([0, T]; \mathbb{R}^d)\) solution of
\[
\frac{d\tilde{p}}{dt} (t) = -A(t)^+(\tilde{p}(t) - \tilde{b}(t)), \forall t \in [0, T], \tilde{p}(T) = \frac{d\tilde{x}}{dt} (\tilde{x}(T)) \in R^d
\]

Where for all \(t \in [0, T]::
\[
A(t) = \frac{df}{dx} (t, \tilde{x}(t), \tilde{u}(t)) \in R^{d \times d}, \tilde{b}(t) = \frac{dg}{dx} (t, \tilde{x}(t), \tilde{u}(t)) \in R^d.
\]
We have: \(\tilde{u}(t) \in arg_{u \in U} min H(t, \tilde{x}(t), \tilde{p}(t), u)\) \(p, p, t \in [0, T]\)

Where the Hamiltonian \(H : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times U \to \mathbb{R}\) verify:
\[
H(t, x, p, u) = p^T f(x, x, u) + g(t, x, u)
\]

A triplet \((\tilde{x}, \tilde{p}, \tilde{u})\) satisfying the above conditions is called an extremal.

Remark:
The PMP here provides only a necessary condition for optimization. It does not say anything about the existence of optimal control and it does not provide a sufficient condition a priori. In practice, we consider the extremes and we sort...

**PMP extension: (The target hit)**

We add the constraint of reaching a target variety $M$ at the instant $t = T / x(T) \in M$.

$M$ is a $C^1$ differential Variety of dimension $0 \leq d_0 \leq d$.

At a point $x_1 \in \mathbb{M}$, the tangent space $T_{x_1}M$ is the set of velocity vectors of the curves drawn on $\mathbb{M}$ passing through $x_1$.

If $d_0' = 0$, $M = \{ x_1 \}$ (point target), and $T_{x_1} M = \{ 0 \}$.

If $d_0' = d$, $M = \{ R^d \}$ (trivial target constraint), and $T_{x_1} M = R^d$.

And we look an optimal control in $U_M := \{ u \in L^1 ([0,T]; U) \mid x_u(T) \in M \}$.

**PMP with target**

If $u \in U_M$ is an optimal control, then by setting $\bar{x} := x_u \in AC ([0,T]; R^d)$ such as

$$\frac{d\bar{x}}{dt} (t) = -\bar{A}(t) \bar{p}(t) - \lambda \bar{B}(t), \forall t \in [0,T]$$

Where: $\bar{A}(t) = \frac{df}{dx} (t, \bar{x}(T), \bar{u}(T)) \in R^{d \times d}$, $\bar{B}(t) = \frac{dg}{dx} (t, \bar{x}(T), \bar{u}(T)) \in R^d$.

And satisfying the condition of transversality in final time $\bar{p}(t) - \lambda \frac{dh}{dx} (\bar{x}(T)) \perp T_{\bar{x}(T)} M$

We have $(p, \lambda) \neq (0,0)$ and: $\bar{u}(t) = arg_{u \in U} min H(t, \bar{x}(t), \bar{p}(t), \lambda, u) \mid p, t \in [0,T]$

Where the Hamiltonian $H : [0,T] \times R^d \times R^d \times R_+ \times U \rightarrow R$ verify:

$$H(t, x, p, \lambda, u) = p^T f (t, x, u) + \lambda g(t, x, u)$$

A quadruplet $(\bar{x}, \bar{p}, \lambda, \bar{u})$ satisfying the above conditions is called an extremal.

**Important comments**

- As $(p, \lambda) \neq (0,0)$, two cases can occur:
  1. $\lambda \neq 0$ : the PMP being invariant by a positive scaling factor on $(p, \lambda)$, we can assume that $\lambda = 1$; we say that the extremal is normal
  2. $\lambda = 0$ : we necessarily have $p \neq 0$; we say that the extremal is abnormal

- When the variety $M$ is of dimension $d_0 < d$, there are more possibilities for $p(T)$, and there may be abnormal extremals.

**Formulation of the problem:**

**Zermelo problem** (with current speed more than Boat speed hypothesis)

Boat crossing a channel of width $\ell$

- The control is the angle $u$ of the boat p.r. to the horizontal axis representing the two banks
- Boat of speed $v$, current of speed $c(y)$
- We suppose that $c(y) > v, \forall y \in [0,1]$ (strong current hypothesis)
We consider three optimal control problems to reach the bank opposite
1. minimize the lateral offset
2. minimize the crossing time
3. reach a point on the opposite bank in minimum time.

Minimization of the lateral offset

The condition of the boat is described by the couple \( X = \begin{pmatrix} x \\ y \end{pmatrix} \) giving the coordinates of the boat.

The trajectory of the boat is governed by the following dynamics:

\[
\begin{align*}
\dot{X}(t) &= f(X(t), u(t)) = \begin{pmatrix} v \cos(u(t)) + c(y(t)) \\ v \sin(u(t)) \end{pmatrix} \\
X(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]

In the lateral offset minimization problem, the criterion involves functions

\( g(t, X, u) \equiv 0 \) and \( h(X) = x \)

And by adding the target constraint \( y(T) = 1 \); the final time is free.

As \( \tilde{A}(t) = \begin{pmatrix} 0 & c'(\tilde{y}(t)) \\ 0 & 0 \end{pmatrix} \) and \( \tilde{B}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), the deputy state \( \tilde{p}(t) = \begin{pmatrix} \tilde{p}_x \\ \tilde{p}_y \end{pmatrix} \) satisfies:

\[
\tilde{p}_x = \text{cst} \quad \text{and} \quad \frac{d\tilde{p}_x}{dt}(t) = -c'(\tilde{y}(t)) \tilde{p}_x
\]

And the transversality condition on \( \tilde{p}(T) \) gives \( \begin{pmatrix} \tilde{p}_x - \lambda \\ \tilde{p}_y \end{pmatrix} \perp \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \tilde{p}_x = \lambda
\]

The Hamiltonian is worth

\[
H(X, p, \lambda, u) = (p_x \cos(u) + p_y \sin(u)) v + p_x c(y)
\]

The minimization condition gives us the optimal control if \( |p| \neq 0 \):

\[
\cos(\tilde{u}) = -\frac{p_x}{|p|} \quad \sin(\tilde{u}) = -\frac{p_y}{|p|}
\]

So, the minimized Hamiltonian is \( H(\tilde{X}(t), \tilde{p}(t), \lambda, \tilde{u}(t)) = -|\tilde{p}(t)| v + p_x c(y(t)) \)

And the transversality condition on \( H \) gives \( -|\tilde{p}(T)| v + p_x c(y(T)) = 0 \)

We deduce that there is no abnormal extremal. Because:

- if \( \lambda = 0 \) than \( \tilde{p}_x = \lambda = 0 \); therefore \( p_y \) is constant and will cancelled at final time (transversality over \( H \)) ⇒ \( p_y \) would also be zero, which is excluded ⇒ so we can assume that \( \tilde{p}_x = \lambda = 1 \)
- therefore, we always have \( \tilde{p} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) so that the optimal control is good such that:

\[
\cos(\tilde{u}(t)) = -\frac{v}{c'(\tilde{y}(t))}, \quad \forall \ t \in [0, T]
\]

The angle of the trajectory in the Cartesian frame is \( \tilde{u}(t) = \frac{v}{z} \).

Comments:

**The minimized Hamiltonian Stationarity:**

- We have \( H(X, p, \lambda, u) = p^t f(X, u) + \lambda g(X, u) \) here

\( g = 0 \)

- the optimal control \( \tilde{u} \) being regular in time, the minimization of the Hamiltonian implies that

\[
\frac{d}{dt} (\tilde{X}(t), \tilde{p}(t), \lambda, \tilde{u}(t)) = 0
\]

- As \( \frac{d\tilde{p}}{dt}(t) = \frac{d}{dt} (\tilde{X}(t), \tilde{p}(t), \lambda, \tilde{u}(t)) \) and \( \frac{d\tilde{p}}{dt}(t) = -\frac{d}{dx} (\tilde{X}(t), \tilde{p}(t), \lambda, \tilde{u}(t)) \)

We will have:

\[
\frac{d}{dt} H(\tilde{X}(t), \tilde{p}(t), \lambda, \tilde{u}(t)) = 0
\]
the minimized Hamiltonian being zero at $T$ (by transversality), we conclude that:

$$H(\bar{X}(t), \bar{P}(t), \lambda, \bar{u}(t)) = 0, \forall \ t \in [0, T]$$

- This allows to show directly that $|\bar{p}(t)| = \frac{c'(\bar{y}(t))}{v}, \forall \ t \in [0, T]$

Achievable set:
Solving the lateral offset problem allows us to determine the set reachable by any control

![Diagram of a control system](image)

2. Minimization of crossing time

The criterion this time involves the functions $g(t, X, u) \equiv 1, h(X) \equiv 0$, we still have the target constraint $y(T) = 1$ and the final time remains free.

The deputy state equations are unchanged ($\bar{p}_x = \text{cst}$ and $\frac{d\bar{p}_x}{dt}(t) = -c'\left(\bar{y}(t)\right)\bar{p}_x$). But the transversality condition on $\bar{p}$ is now $\bar{p}_x(T) = 0$ hence $\bar{p}_x \equiv 0$ So $\bar{p}_y = \text{cst}e$

The Hamiltonian is worth $H(\bar{X}, \bar{P}, \lambda, u) = \left(p_x \cos(u) + p_y \sin(u)\right) v + p_x c(y) + \lambda$, and:

- $H(\bar{X}, \bar{P}, \lambda, u) = p_y \sin(u) v + \lambda$ because $\bar{p}_x \equiv 0$
- $\bar{p}_y \neq 0$, otherwise, the transversality cond on $H$ would give $\lambda = 0$
- The optimal command is therefore $\sin(\bar{u}(t)) = 1 \text{ i.e. } \bar{u}(t) = \frac{\pi}{2}$

Note: Never sail against the current if you want to reach the opposite shore as quickly as possible.

3. Reach a target in minimal time:

The target is $\left(\frac{X_1}{1}\right)$ where $x_1$ is located downstream of the point of minimum offset.

We consider $g \equiv 1, h \equiv 0$. The deputy state equations are still unchanged ($\bar{p}_x = \text{cst}$ and $\frac{d\bar{p}_x}{dt}(t) = -c'\left(\bar{y}(t)\right)\bar{p}_x$).

But the transversality condition on $\bar{p}$ become trivial.

The Hamiltonian is $H(\bar{X}, \bar{P}, \lambda, u) = \left(p_x \cos(u) + p_y \sin(u)\right) v + p_x c(y) + \lambda$, and the minimized Hamiltonian is $-|\bar{p}(t)| v + p_x c(y(t)) + \lambda \equiv 0, \forall \ t \in [0, T]$.

- The Abnormal extremal ($\lambda = 0$);
  we get again $\cos(\bar{u}(t)) = -\frac{v}{c'(t)}$, this is possible if $x_1$ is the abscissa of the minimum offset point.
- The Normal extremal ($\lambda = 1$):
  using the minimized Hamiltonian and $\cos(\bar{u}(t)) = -\frac{p_x}{|\bar{p}(t)|}$ we will have:

$$\cos(\bar{u}(t)) = \frac{p_x v}{1 - p_x c'(t)}$$

provided that $\bar{p}_x \in \left[\frac{1}{v + \max_{y \in [0,1]} c'(y)}\right]$.

we obtain a family of curves with one parameter, such as when $\bar{p}_x \to -\infty$ we tend towards the abnormal extremal.

The value $\bar{p}_x = 0$ corresponds to the crossing in minimum time.

Bibliography: