ARTICLE HISTORY: Received: September 13, 2021 Accepted: November 5, 2021 Published: November 13, 2021

EXAMPLE OF APPLICATION OF THE PONTRYAGIN'S MINIMUM PRINCIPLE "PMP EXTENSION": ZERMELO PROBLEM (WITH CURRENT SPEED MORE THAN BOAT SPEED HYPOTHESIS)

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ПРИМЕР ПРИМЕНЕНИЯ ПРИНЦИПА МИНИМУМА ПОНТРЯГИНА «РАСШИРЕНИЕ PMP»:ЗАДАЧА ZERMELO (ПРИ ГИПОТЕЗЕ СКОРОСТЬ ТЕЧЕНИЯ БОЛЬШЕ, ЧЕМ СКОРОСТИ ЛОДКИ)

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Abstract. We present an example of application covering several cases using the extension of the Pontryaguine minimum principle (PMP) in the case where we add a constraint on reaching a target variety at the final time: the Zermelo problem with current speed more than Boat speed hypothesis, where we consider a boat crossing a channel under a strong current and where we try to reach the opposite bank by minimizing the lateral offset or by minimizing the crossing time

Аннотация. Мы представляем пример приложения, охватывающий несколько случаев, использующих расширение принципа минимума Понтрягуйна (PMP) в случае, когда мы добавляем ограничение на достижение целевого разнообразия в последний момент: проблема Zermelo с скоростью течения больше, чем скорости лодки, где мы рассматриваем лодку, пересекающую канал под сильным течением, и где мы пытаемся достичь противоположного берега, минимизируя боковое смещение или минимизируя время перехода

Key words: Zermelo problem– Hamiltonian - optimal control – Minimization - Non-linear control systems

Ключевые слова: задача Zermelo - гамильтониан - оптимальное управление - минимизация - нелинейные системы управления.

Definition: Non-linear control systems

We consider the non-linear control system

$\dot{x}_u(t) = f\left(t, x_u(t), u(t)\right), \forall t \in [0,T], x_u(0) = x_0$

- $x_u : [0,T] \to R^d, T > 0\ fixed, x_0 \in R^d$ fixed
- $u : [0,T] \to U \subset R^k$, a closed non-empty sub-assembly
- $f : [0,T] \times R^d \times U \to R^d$

We are looking for an optimal control $\bar{u} \in U = L^1([0,T];U)$ which minimizes the criterion $J(u) = \int_0^T g(t, x_u(t), u(t))\, dt + h(x_u(T))$

With $g : [0,T] \times R^d \times U \to R^d$ and $h : R^d \to R$

With the usual known hypotheses on f, g and h

The theorem of the Pontryagin's minimum principle PMP:

If $\bar{u} \in U$ is an optimal control, then by noting $\bar{x} = x_{\bar{u}} \in AC\ ([0,T]; R^d)$ the trajectory associated with the control $\bar{u}$ and by defining the deputy state $\bar{p} \in AC\ ([0,T]; R^d)$ solution of $\frac{d\bar{p}}{dt}(t) = -\bar{A}(t)^{\dagger}\bar{p}(t) - \bar{B}(t), \forall t \in [0,T], \bar{p}(T) = \frac{dh}{dx}\left(\bar{x}(T)\right) \in R^d$

Where for all $t \in [0,T]$:

$$\bar{A}(t) = \frac{df}{dx}\left(t, \bar{x}(T), \bar{u}(T)\right) \in R^{d\times d}, \bar{B}(t) = \frac{dg}{dx}\left(t, \bar{x}(T), \bar{u}(T)\right) \in R^d,$$

We have : $\bar{u}(t) \in arg_{v \in U} min\, H(t, \bar{x}(t), \bar{p}(t), v)\ p.p. t \in [0,T]$

Where the Hamiltonian $H : [0,T] \times R^d \times R^d \times U \to R$ verify:

$$H(t,x,p,u) = p^{\dagger} f\,(t,x,u) + g(t,x,u)$$

A triplet $(\bar{x}, \bar{p}, \bar{u})$ satisfying the above conditions is called an extremal.

Remark:

The PMP here provides only a necessary condition for optimization. It does not say anything about the existence of optimal control and it does not provide a sufficient condition a priori. In practice, we consider the extremes and we sort...

PMP extension: (The target hit)

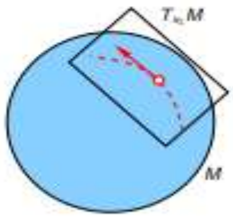
We add the constraint of reaching a target variety M at the instant $t = T \,/\, x(T) \in M$

M Is a C^1 differential Variety of dimension $0 \le d_0 \le d$

At a point $x_1 \in M$, the tangent space $T_{x_1}M$ is the set of velocity vectors of the curves drawn on M passing through x_1

If $d'_0 = 0$, $M = \{x_1\}$ (point target), and $T_{x_1}M = \{0\}$

If $d'_0 = d$, $M = \{R^d\}$ (trivial target constraint), and $T_{x_1}M = R^d$



And we look an optimal control in $U_M := \{u \in L^1([0,T];U) \mid x_u(T) \in M\}$

PMP with target

If $u \in U_M$ is an optimal control, then by setting $\bar{x} := x_{\bar{u}} \in AC([0,T];R^d)$, there exists $\lambda \in R_+$ such as by defining the deputy state $\bar{p} \in AC([0,T];R^d)$ such as

$\frac{d\bar{p}}{dt}(t) = -\bar{A}(t)^{\dagger}\bar{p}(t) - \lambda\,\bar{B}(t), \forall t \in [0,T]$

Where: $\bar{A}(t) = \frac{df}{dx}\left(t, \bar{x}(T), \bar{u}(T)\right) \in R^{d\times d}$,$\bar{B}(t) = \frac{dg}{dx}\left(t, \bar{x}(T), \bar{u}(T)\right) \in R^d$,

And satisfying the condition of transversality in final time $\bar{p}(t) - \lambda\,\frac{dh}{dx}\left(\bar{x}(T)\right) \perp T_{\bar{x}(T)}M$

We have $(p,\lambda) \neq (0,0)$ and : $\bar{u}(t) = arg_{v\in U} min\, H(t, \bar{x}(t), \bar{p}(t), \lambda, v)$ $p.p.\, t \in [0,T]$

Where the Hamiltonian $H : [0,T] \times R^d \times R^d \times \boldsymbol{R}_+ \times U \to R$ verify:

$$H(t,x,p,\lambda,u) = p^{\dagger} f\,(t,x,u) + \lambda\, g(t,x,u)$$

A quadruplet$(\bar{x}, \bar{p}, \lambda, \bar{u})$ satisfying the above conditions is called an extremal

Important comments

- As $(p,\lambda) \neq (0,0)$, two cases can occur:

1. $\lambda \neq 0$: the PMP being invariant by a positive scaling factor on (p,λ), we can assume that $\lambda = 1$; we say that the extremal is normal
2. $\lambda = 0$: we necessarily have $p \neq 0$; we say that the extremal is abnormal

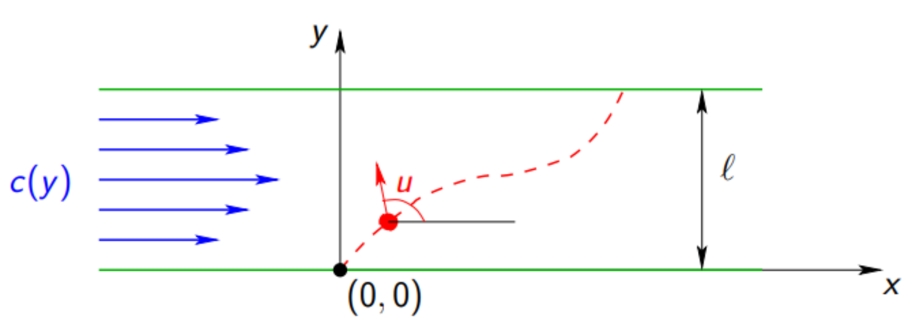
- When the variety M is of dimension $d_0 < d$, there are more possibilities for $p\,(T)$, and there may be abnormal extremals.

<u>**Formulation of the problem:**</u>

Zermelo problem (with current speed more than Boat speed hypothesis)

Boat crossing a channel of width l

- The control is the angle u of the boat p.r. to the horizontal axis representing the two banks
- Boat of speed v, current of speed $c\,(y)$
- we suppose that $c\,(y) > v, \forall y \in [0,l]$ (strong current hypothesis)



We consider three optimal control problems to reach the bank opposite

1. minimize the lateral offset
2. minimize the crossing time
3. reach a point on the opposite bank in minimum time.

1. Minimization of the lateral offset

The condition of the boat is described by the couple $X = \binom{x}{y}$ giving the coordinates of the boat

The trajectory of the boat is governed by the following dynamics:

$$\begin{cases} \dot{X}(t) = f\left(X(t), u(t)\right) = \begin{pmatrix} v\cos(u(t)) + c(y(t)) \\ v\sin(u(t)) \end{pmatrix} \\ X(0) = \binom{0}{0} \end{cases}$$

In the lateral offset minimization problem, the criterion involves functions

$$g(t, X, u) \equiv 0 \; and \; h(X) = x$$

And by adding the target constraint $y\,(T) = l$; the final time is free

As $\bar{A}(t) = \begin{pmatrix} 0 & c'(\bar{y}(t)) \\ 0 & 0 \end{pmatrix}$ and $\bar{B}(t) = \binom{0}{0}$, the deputy state $\bar{p}(t) = \binom{\bar{p}_x}{\bar{p}_y}$ satisfied:

$$\bar{p}_x = \text{cst and } \frac{d\bar{p}_y}{dt}(t) = -c'\left(\bar{y}(t)\right)\bar{p}_x$$

And the transversality condition on $\bar{p}(T)$ gives $\begin{pmatrix} \bar{p}_x - \lambda \\ \bar{p}_y(T) \end{pmatrix} \perp \binom{1}{0} \Rightarrow \bar{p}_x = \lambda$

The Hamiltonian is worth

$H(X, p, \lambda, u) = \left(p_x\,cos(u) + p_y\,sin(u)\right)v + p_x c(y)$

The minimization condition gives us the optimal control if $|p| \neq 0$:

$cos(\bar{u}) = -\frac{\overline{p_x}}{|\bar{p}|}$, $sin(\bar{u}) = -\frac{\overline{p_y}}{|\bar{p}|}$

So, the minimized Hamiltonian is $H(\bar{X}(t), \bar{p}(t), \lambda, \bar{u}(t)) = -|\bar{p}(t)|v + p_x c(y(t))$

And the transversality condition on H gives $-|\bar{p}(T)|v + p_x c\left(y(T)\right) = 0$

We deduce that there is no abnormal extremal Because:

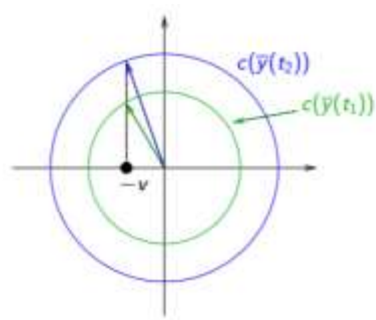
- if $\lambda = 0$ than $\bar{p}_x = \lambda = 0$; therefore p_y is constant and will cancelled at final time (transversality over H) ⇒ p_y would also be zero, which is excluded ⇒ so we can assume that $\bar{p}_x = \lambda = 1$
- therefore, we always have $\bar{p} \neq \binom{0}{0}$ so that the optimal control is good such that: $cos(\bar{u}) = -\frac{\overline{p_x}}{|\bar{p}|}$, $sin(\bar{u}) = -\frac{\overline{p_y}}{|\bar{p}|}$

We have $|\bar{p}(t)| = \frac{c(\bar{y}(t))}{v}, \forall\, t \in [0, T]$; this relation is satisfied in T (transversality on H). by deriving in relation to time, as $|\bar{p}| = (1 + \overline{p_y}^2)^{1/2}$ becomes :

$$\frac{d|\bar{p}|}{dt} = \frac{\overline{p_y}}{|\bar{p}|}\frac{d\overline{p_y}}{dt} = sin(\bar{u})c'y(t) = \frac{1}{v}\frac{d\,\bar{y}}{dt}c'\bar{y}(t) = \frac{1}{v}\frac{d\bar{c}(\bar{y})}{dt}$$

In conclusion, the optimal control is written as feedback

$cos(\bar{u}(t)) = -\frac{v}{c\bar{y}(t)}, \forall\, t \in [0, T]$



The angle of the trajectory in the Cartesian frame is $\bar{u}(t) - \frac{\pi}{2}$.

Comments:

the minimized Hamiltonian Stationarity:

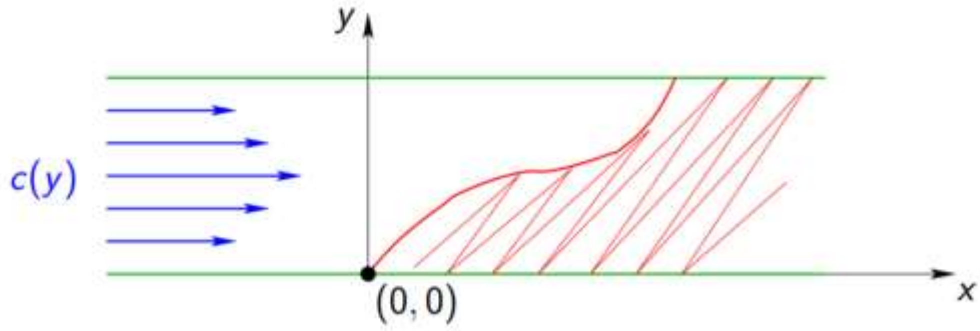
- We have $H(X, p, \lambda, u) = p^{\dagger}f\,(X, u) + \lambda g(X, u)$ here $g = 0$
- the optimal control $\bar{u}$ being regular in time, the minimization of the Hamiltonian implies that $\frac{dH}{du}(\bar{X}(t), \bar{p}(t), \lambda, \bar{u}(t)) = 0$
- As $\frac{d\bar{X}}{dt}(t) = \frac{dH}{dp}(\bar{X}(t), \bar{p}(t), \lambda, \bar{u}(t))$ and $\frac{d\bar{p}}{dt}(t) = -\frac{dH}{dx}(\bar{X}(t), \bar{p}(t), \lambda, \bar{u}(t))$ We will have: $\frac{d}{dt}H(\bar{X}(t), \bar{p}(t), \lambda, \bar{u}(t)) = 0$

the minimized Hamiltonian being zero at T (by transversality), we conclude that: $H(\bar{X}(t), \bar{p}(t), \lambda, \bar{u}(t)) = 0, \forall t \in [0, T]$

- This allows to show directly that $|\bar{p}(t)| = \frac{c(\bar{y}(t))}{v}, \forall t \in [0, T]$

Achievable set:

Solving the lateral offset problem allows us to determine the set reachable by any control



2. Minimization of crossing time

The criterion this time involves the functions $g(t, X, u) \equiv 1, h(X) \equiv 0$, we still have the target constraint $y(T) = l$ and the final time remains free.

The deputy state equations are unchanged ($\bar{p}_x = \text{cst}$ and $\frac{d\bar{p}_y}{dt}(t) = -c'(\bar{y}(t))\,\bar{p}_x$). But the transversality condition on $\bar{p}$ is now $\bar{p}_x(T) = 0$ hence $\bar{p}_x \equiv 0$ So $\bar{p}_y = cste$

The Hamiltonian is worth $H(X, p, \lambda, u) = (p_x \cos(u) + p_y \sin(u))\, v + p_x c(y) + \lambda$, and:

- $H(\bar{X}, \bar{p}, \lambda, u) = p_y \sin(u)\; v + \lambda$ because $\bar{p}_x \equiv 0$
- $\bar{p}_y \neq 0$, otherwise, the transversality cond on H would give $\lambda = 0$
- The optimal command is therefore $\sin(\bar{u}) = 1$ i.e. $\bar{u}(t) = \frac{\pi}{2}$

Note: Never sail against the current if you want to reach the opposite shore as quickly as possible.

3. Reach a target in minimal time:

The target is $\binom{x_1}{l}$ where x_1 is located downstream of the point of minimum offset.

We consider $g \equiv 1, h \equiv 0$. The deputy state equations are still unchanged ($\bar{p}_x = \text{cst}$ and $\frac{d\bar{p}_y}{dt}(t) = -c'(\bar{y}(t))\,\bar{p}_x$). But the transversality condition on $\bar{p}$ become trivial.

The Hamiltonian is $H(X, p, \lambda, u) = (p_x \cos(u) + p_y \sin(u))\, v + p_x c(y) + \lambda$, and the minimized Hamiltonian is $-|\bar{p}(t)|v + p_x c(y(t)) + \lambda \equiv 0, \forall t \in [0, T]$.

- The Abnormal extremal ($\lambda = 0$):

we get again $\cos(\bar{u}(t)) = -\frac{v}{c\bar{y}(t)}$, this is possible if x_1 is the abscissa of the minimum offset point.

- The Normal extremal ($\lambda = 1$):

using the minimized Hamiltonian and $\cos(\bar{u}(t)) = -\frac{\bar{p}_x}{|\bar{p}(t)|}$ we will have:

$\cos(\bar{u}(t)) = \frac{\bar{p}_x v}{1 - \bar{p}_x c\bar{y}(t)}$ provided that $\bar{p}_x \in \left]-\infty, \frac{1}{v + \max_{y \in [0,l[} c(y)}\right[$

we obtain a family of curves with one parameter, such as when $\bar{p}_x \to -\infty$ we tend towards the abnormal extremal. The value $\bar{p}_x = 0$ corresponds to the crossing in minimum time.

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