ZIGMUND TYPE ESTIMATES FOR MIXED FRACTIONAL INTEGRALS OF THE VOLTERRA CONVOLUTION TYPE

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Abstract. Non-weight Zygmund-type estimates are obtained for mixed fractional integrals of the Volterra convolution type for a function of two variables defined by a mixed modulus of continuity.

Аннотация. Получены оценки типа Зигмунда для смешанных дробных интегралов типа вольтеровской свертки в обобщенном Гельдервском пространстве для двух переменных определяемых смешанным модулем непрерывности.

Keywords: mixed fractional integral, mixed modulus of continuity, Zygmund type estimate, two variable function, Volterra convolution type.

Ключевые слова: функции двух переменных, смешанный дробный интеграл, смешанный модуль непрерывность, смешанный дробный интеграл типа вольтеровской свертки, оценки типа Зигмунда.

Introduction

An important stage in the study of fractional integro-differentiation of functions from generalized Hölder spaces is the obtaining of Zygmund-type estimates, i.e. estimate of the modulus of continuity of a fractional integral through the modulus of continuity of the original function. A similar problem can be considered completely solved for the Hölder space of functions of one variable and power weights (see [2], [6], see also [1]), as well as for the Hölder space of functions of two variables and power weights (see [3]-[5]). Obtaining an estimate of the Zygmund type for mixed fractional integrals with arbitrary kernels and has not been studied.

The main focus of the work is to obtain an estimate of the type of Zygmund majorizing a mixed modulus of continuity $\omega\left(\rho\widetilde{K}\phi;h,\eta\right)$ mixed fractional integral with a Volterra convolution type of weight integral constructions from a mixed modulus of continuity $\omega(\rho\phi;h,\eta)$ its density $\phi(x,y)$ with weight $\rho(x,y)$. These Zygmund type bounds and action theorems directly affect the nature of the improvement of the modulus of continuity by mixed fractional integration of the Volterra convolution type:

$$\left(\widetilde{K}\phi\right)(x,y)=\int_0^x \int_0^y k(x-t)k(y-s)\phi(t,s)dtds, \qquad (1)$$

here we consider the degenerate kernels, as well as each $k(x)$ and $k(y)$ assumed to be close in some sense to a power function.

In this paper, we deal with arbitrary kernels, i.e. not necessarily power. We will consider the operator (1) in a rectangle $Q=\{(x,y):0<x<b,0<y<d\}$.

Preliminary

1. One-dimensional statements. In this section, we present some well-known results and notation that we will need to present the issues under consideration (see [5]).

Everywhere in the results through $C,C_1,C_2,\ldots$ we denote absolute constants that can have different meanings in different cases.

Definition 1. We say that $k(x)\in V_\lambda,\lambda>0,k(x)>0$, if $k(x)\in[0,l]$ and satisfies the conditions:

1)$k(x)\neq 0,x^\lambda k(x)$ - almost increases and $x^\lambda k(x)|x=0=0$;

2)$\exists\varepsilon>0$, $0<\varepsilon<\lambda$, that $x^{\lambda-\varepsilon}k(x)$ - almost decreases;

$$3)\exists C\text{, that } \left|\frac{k(x_1)-k(x_2)}{x_1-x_2}\right|\le C\,\frac{k(x*)}{x*}\text{, if } x*=max(x_1,x_2). \qquad (2)$$

Definition 2. Let given bounded on $[a,b]$ function $\phi(x)$. Under modulus of continuity$\phi(x)$ understood expression

$$\sup_{h\in[0,\delta]}|\phi(x+h)-\phi(x)|=\omega(\phi;\delta),\ 0<\delta\le b-a.$$

Definition 3. We denote by Φ^1 function class $\omega(\delta)\in(0,b-a]$ and satisfying the conditions

1) $\boldsymbol{\omega(\delta)>0}$ ***in (0, b–a],*** $\boldsymbol{\lim_{\delta\to 0}\omega(\delta)=0}$***;***

2) $\boldsymbol{\omega(\delta)\uparrow}$ ***in (0, b–a];***

3) $\boldsymbol{\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)}$***.***

Below in the estimates we need inequalities:

1) if $\omega(\phi;h)$ is modulus continuity, then we have

$$x_2\omega(\phi;x_1)\le Cx_1\omega(\phi;x_2),\ x_2\le x_1; \qquad (3)$$

2) if $k(x)\in V_\lambda$, then $x_1^\lambda k(x_1)\le Cx_2^\lambda k(x_2)$ and $\exists 0<\varepsilon<\lambda$,

$$x_2^{\lambda-\varepsilon}k(x_2)\le Cx_1^{\lambda-\varepsilon}k(x_1),\ x_1\le x_2; \qquad (4)$$

Lemma 1. Let $k(x)\in V_\lambda,\ \lambda>0$ and $\omega(x)\ge 0$ and almost increases, then for any $0<x<l/2$, rightly

$$\omega(x)k(x) \le C\int_x^l t^{-1}\omega(t)k(t)dt. \tag{5}$$

Consider a one-dimensional integral operator of the Voltero type convolutions

$(K\phi)(x) = \int_0^x k(x-t)\phi(t)dt, \quad 0 < x < b.$ (6)

Theorem 1. Let $k(x) \in V_\lambda, 0 < \lambda < 1$ и $\phi(x) \in C([0,b]), \phi(0) = 0$. Then for integral (4), the following estimate is valid

$$\omega(K\phi, h) \le C\left(hk(h)\omega(\phi, h) + h\int_h^b \frac{k(t)\omega(\phi,t)}{t}dt\right) \tag{7}$$

Proof. The integral (4) will be presented in the form

$$(K\phi)(x) = \phi(0)\int_0^x k(x-t)dt + \int_0^x k(x-t)[\phi(t) - \phi(0)]dt.$$

Since $\phi(0) = 0$, then we have $f(x) = \int_a^x g(t)k(t-a)dt$, где $\phi(t) - \phi(0) = g(t)$.

Let $h > 0$, $x, x+h \in [0,b]$. Consider the difference

$$|f(x+h) - f(x)| \le C\left|\int_{-h}^0 k(h+t)[g(x-t) - g(x)]dt + \right.$$

$$\left. + \int_0^x (k(h+t) - k(t))[g(x-t) - g(x)]dt + g(x)\int_x^{x+h} k(t)dt\right| = A_1 + A_2 + A_3. \tag{8}$$

We estimate A_1. Using inequality (3) and by definition 1, we have

$$A_1 \le \int_0^h \omega(\phi,t)k(h-t)dt \le C\omega(\phi,h)k(h)\int_0^h \left(\frac{h}{h-t}\right)^\lambda dt \le Chk(h)\omega(\phi,h). \tag{9}$$

We estimate A_2. Using inequality (2), we have

$$A_2 \le C\int_0^x \frac{k(h+t)}{h+t}\omega(\phi;t)dt.$$

Here we distinguish two cases: 1) $h \ge x$ and 2) $h < x$. In the first case

$$A_2 \le Ch^{1+\lambda-\varepsilon}k(h)\int_0^x \frac{\omega(\phi;t)}{(h+t)^{1+\lambda-\varepsilon}}dt = Chk(h)\int_0^{\frac{x}{h}} \omega(\phi,ht)\frac{dt}{(1+t)^{1+\lambda-\varepsilon}} \le$$

$$\le Chk(h)\int_0^1 \frac{\omega(\phi,ht)}{(1+t)^{1+\lambda-\varepsilon}}dt \le Chk(h)\omega(\phi,h). \tag{10}$$

In the second case A_2, it can be represented as the sum of two terms, i.e.$A_2 \le \int_0^h + \int_h^{x-a} = A_2' + A_2''$, для справедлива оценка (10). А для, имеем (for A'_2 is rightly estimate (10). And for A''_2, we have

$$A_2'' \le Ch\int_h^x \omega(\phi,t)\frac{k(t)}{t}dt \le Ch\int_h^b \frac{\omega(\phi,t)}{t}k(t)dt.$$

Finally, we estimateA_3. By $x \le h$:

$$A_3 \le C\omega(\phi,x)k(x+h)(x+h)^\lambda\int_x^{x+h} \frac{dt}{t^\lambda} \le C\omega(\phi,h)k(h)h.$$

If $x > h$, then using Lemma 1 and we have

$$A_3 \le C\omega(\phi,x)hk(x) \le Ch\int_x^b \frac{k(t)}{t}\omega(\phi;t)dt \le Ch\int_x^b \frac{k(t)}{t}\omega(\phi;t)dt.$$

Collecting all estimates for $A_1,\ A_2,\ A_3$, we get (7).

Let a continuous function $\phi(x,y)$ be defined in R^2. We introduce the necessary notation

$$\left(\overset{1,0}{\Delta}_h\phi\right)(x,y) = \phi(x+h,y) - \phi(x,y), \left(\overset{0,1}{\Delta}_\eta\phi\right)(x,y) = \phi(x,y+\eta) - \phi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta}\phi\right)(x,y) = \phi(x+h,y+\eta) - \phi(x,y+\eta) - \phi(x+h,y) + \phi(x,y),$$

and

$$\phi(x+h,y+\eta) = \left(\overset{1,1}{\Delta}_{h,\eta}\phi\right)(x,y) + \left(\overset{0,1}{\Delta}_\eta\phi\right)(x,y) + \left(\overset{1,0}{\Delta}_h\phi\right)(x,y) + \phi(x,y). \tag{11}$$

Now we introduce the following characteristics:

1) Private modules of continuity

$$\overset{1,0}{\omega}(\phi;\delta,0) = \sup_y \sup_{0\le h\le\delta}\left|\left(\overset{1,0}{\Delta}_h\phi\right)(x,y)\right| \text{ и } \overset{0,1}{\omega}(\phi;0,\sigma) = \sup_x \sup_{0\le\eta\le\sigma}\left|\left(\overset{0,1}{\Delta}_\eta\phi\right)(x,y)\right|;$$

2) Mixed modulus continuity of order 1.1

$$\overset{1,1}{\omega}(\phi;\delta,\sigma) = \sup_{x,y}\sup_{\substack{0\le h\le\delta\\0\le\eta\le\sigma}}\left|\left(\overset{1,1}{\Delta}_{h,\eta}\phi\right)(x,y)\right|, \text{ где } 0 < \delta \le b, 0 < \sigma \le d.$$

It follows from the definition $\overset{1,1}{\omega}(\phi;\delta,\sigma)$ that this function belongs in each variable Φ^1. In addition, we note that there is an inequality

$$\overset{1,1}{\omega}(\phi;\delta,\sigma) \le 2\,min\left\{\overset{1,0}{\omega}(\phi;\delta,0), \overset{0,1}{\omega}(\phi;0,\sigma)\right\}. \quad (12)$$

Main result

In this section, we generalize Theorem 1 to the case $k(x,y) = k_1(x)k_2(y)$.

Theorem 2. Let $k_1(x),\ k(y) \in V_\lambda,\ 0 < \lambda < 1,\ \phi(x,y) \in C(Q)$ and $\phi(x,y)|_{x=0,y=0} = 0$. Then Zygmund type estimates are valid

$$\overset{1,0}{\omega}(\widetilde{K}\phi;h,0) \le C_1\left[hk(h)\overset{1,1}{\omega}(\phi;h,d) + h\int_h^b \frac{k(t)}{t}\overset{1,1}{\omega}(\phi;t,d)dt\right], \quad (13)$$

$$\overset{0,1}{\omega}(\widetilde{K}\phi;0,\eta) \le C_2\left[\eta k(\eta)\overset{1,1}{\omega}(\phi;b,\eta) + \eta\int_\eta^d \frac{k(s)}{s}\overset{1,1}{\omega}(\phi;b,s)ds\right], \quad (14)$$

$$\overset{1,1}{\omega}(\widetilde{K}\phi;h,\eta) \le C_3\Bigg[h\eta k(h)k(\eta)\overset{1,1}{\omega}(\phi;h,\eta) + h\eta k(\eta)\int_h^b \frac{k(t)}{t}\overset{1,1}{\omega}(\phi;t,\eta)dt +$$
$$+ hk(h)\eta\int_\eta^d \frac{k(s)}{s}\overset{1,1}{\omega}(\phi;h,s)ds + h\eta\int_h^b \int_\eta^d \frac{k(t)k(s)}{ts}\overset{1,1}{\omega}(\phi;t,s)dtds\Bigg]. \quad (15)$$

Proof. Using (11), (1) we will present in the form

$$(\widetilde{K}\phi)(x,y) = \phi(0,0)\int_0^x\int_0^y k(x-t)k(y-s)dtds + \int_0^x\int_0^y k(x-t)k(y-s)\left(\overset{1,1}{\Delta}_{t,s}\phi\right)(0,0)dtds$$
$$+ + \int_0^x\int_0^y k(x-t)k(y-s)[\phi(t,0)-\phi(0,0)]dtds + \int_0^x\int_0^y k(x-t)k(y$$
$$-s)[\phi(0,s)-\phi(0,0)]dtds.$$

Since $\phi(x,y)|_{x=0,y=0} = 0$, then we have

$$f(x,y) = \int_0^x\int_0^y k(x-t)k(y-s)\left(\overset{1,1}{\Delta}_{t,s}\phi\right)(0,0)dtds = \int_0^x\int_0^y g(t,s)k(x-t)k(y-s)dtds.$$

Let $h > 0$, $x, x+h \in [0,b]$. Consider the difference

$$[f(x+h,y)-f(x,y)] = \int_{-h}^0\int_0^y k(h+t)k(s)[g(x-t,y-s)-g(x,y-s)]dtds +$$
$$+ \int_0^x\int_0^y [k(h+t)-k(t)]k(s)[g(x-t,y-s)-g(x,y-s)]dtds +$$
$$+\left(\int_{-h}^x k(h+t)dt - \int_0^x k(t)dt\right)\int_0^y g(x,y-s)ds. \quad (16)$$

Fair inequality

$$|f(x+h,y)-f(x,y)| \le C\Bigg(\int_0^h\int_0^y k(h-t)k(s)\overset{1,1}{\omega}(\phi;t,y-s)dtds +$$
$$+ \int_0^x\int_0^y |k(h+t)-k(t)|k(s)\overset{1,1}{\omega}(\phi;t,y-s) + \int_x^{x+h}\int_0^y k(t)k(s)\overset{1,1}{\omega}(\phi;t,y-s)\,dtds\Bigg) \le$$
$$\le C_1\Bigg(\int_0^h k(h-t)\overset{1,1}{\omega}(\phi;t,d)dt + \int_0^x |k(x+h)-k(t)|\overset{1,1}{\omega}(\phi;t,d)dt +$$
$$+ \int_x^{x+h}|k(x+h)-k(x)|\overset{1,1}{\omega}(\phi;t,d)dt\Bigg).$$

Using the estimates for A_1, A_2, A_3 in the proof of Theorem 1, one can easily verify the inequality (13).

Having made a symmetric permutation (16), we can similarly get (14).

Let $h,\eta > 0$, $\forall x, x+h \in [0,b]$, $\forall y, y+\eta \in [0,d]$. Consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\eta}f(x,y)\right) = \int_{-h}^0\int_{-\eta}^0 \left(\overset{1,1}{\Delta}_{-t,-s}g\right)(x,y)k(t+h)k(s+\eta)dtds +$$
$$+ \int_0^x\int_0^y \left(\overset{1,1}{\Delta}_{-t,-s}g\right)(x,y)[k(h+t)-k(t)][k(\eta+s)-k(s)]dtds +$$
$$+g(x,y)\int_x^{x+h}\int_y^{y+\eta} k(t)k(s)dtds + \int_{-h}^0\int_0^y \left(\overset{1,1}{\Delta}_{-t,-s}g\right)(x,y)k(h+t)[k(\eta+s)-k(s)]dtds +$$
$$+ \int_0^x\int_{-\eta}^0 \left(\overset{1,1}{\Delta}_{-t,-s}g\right)(x,y)[k(h+t)-k(t)]k(s+\eta)dtds +$$
$$+ \int_x^{x+h}\int_{-\eta}^0 [g(x,y-s)-g(x,y)]k(t)k(\eta+s)dtds +$$
$$+ \int_{-h}^0\int_y^{y+\eta} [g(x,y-s)-g(x,y)]k(t+h)k(s)dtds +$$

$$+\int_{x}^{x+h}\int_{0}^{y}[g(x,y-s)-g(x,y)]k(t)[k(\tau+s)-k(s)]dtds+$$
$$+\int_{0}^{x}\int_{y}^{y+\eta}[g(x-t,y)-g(x,y)][k(t+h)-k(t)]k(s)dtds.$$

Fairness inequality

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}f\right)(x,y)\right|\le C\left\{\int_{0}^{h}\int_{0}^{\eta}\overset{1,1}{\omega}(\phi;t,s)k(h-t)k(\eta-s)dtds++\int_{0}^{x}\int_{0}^{y}\overset{1,1}{\omega}(\phi;t,s)|k(h+t)-k(t)||k(\eta+s)-k(s)|dtds+\right.$$
$$+\overset{1,1}{\omega}(\phi;x,y)\int_{x}^{x+h}\int_{y}^{y+\eta}k(t)k(s)dtds+\int_{0}^{h}\int_{0}^{y}\overset{1,1}{\omega}(\phi;t,s)k(h-t)|k(\eta+s)-k(s)|dtds+$$
$$+\int_{0}^{x}\int_{0}^{\eta}\overset{1,1}{\omega}(\phi;t,s)|k(h+t)-k(t)|k(\eta-s)dtds+\int_{x}^{x+h}\int_{0}^{\eta}\overset{1,1}{\omega}(\phi;x,s)k(t)k(\eta-s)dtd+$$
$$+\int_{0}^{h}\int_{y}^{y+\eta}\overset{1,1}{\omega}(\phi;t,y)k(h-t)k(s)dtds+\int_{x}^{x+h}\int_{0}^{y}\overset{1,1}{\omega}(\phi;x,\tau)k(t)|k(\eta+s)-k(s)|dtd+$$
$$\left.+\int_{y}^{y+\eta}\overset{1,1}{\omega}(\phi;t,y)|k(t+h)-k(t)|k(s)dtds\right\}.$$

Estimating the obtained terms in the standard way, we arrive at inequality (15).

References

S.G. Samko, A.A. Kilbas and O.I. Marichev. Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach. Sci. Publ., N. York - London, 1993, 1012 pp. (Russian Ed. - Fractional Interals and Derivatives and Some of Their Applications, Nauka i Tekhnika, Minsk, 1987).

Kh.M.Murdaev. The estimate of the modulus of continuity of the integrals and derivaties of fractional order (Russian). Grozniy, Dep. In VINITI 14.06.85, no 4209, 16 p.

T. Mamatov. Weighted Zygmund estimates for mixed fractional integration. Case Studies Journal ISSN (2305-509X) – Volume 7, Issue 5–May-2018.

T. Mamatov. Mixed Fractional Integration In Mixed Weighted Generalized Hölder Spaces. Case Studies Journal ISSN (2305-509X) – Volume 7, Issue 6–June-2018.

T.Mamatov, S.Samko. Mixed fractional integration operators in mixed weighted Hölder spaces. FCAA. Bulgaria. Vol.13, Num. 3. 2010, p. 245-259.

S. G . Samko, Z. U. Musalaeva. Fractional type operators in weighted generalized Hölder spaces. Proc. of Georgian Acad. Sciencens Math. 1993. Vol 1. P. 601-626.